Local-Lagrangian Quantum Field Theory of Electric and Magnetic Charges*

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We present a local Lagrangian density, depending on a pair of four-potentials A and B, and charged fields ψ_n with electric and magnetic charges e_n and g_n. The resulting local Lagrangian field equations are equivalent to Maxwell's and Dirac's equations. The Lagrangian depends on a fixed four-vector, so manifest isotropy is lost and is regained only for quantized values of $(e_ng_m-g_ne_m)$. This condition results from the requirement that the representation of the Poincaré Lie algebra which results from Poincaré invariance, integrate to a representation of the finite Poincaré group. The finite Lorentz transformation laws of A, B, and ψ_n are presented here for the first time. The familiar apparatus of Lagrangian field theory is applied to yield directly the canonical commutation relations, the energy-momentum tensor, and Feynman's rules.

I. INTRODUCTION

In this paper we present a quantum field theory of electrically and magnetically charged particles that is constructed from a local Lagrangian density which yields local field equations. The first quantum field theory for these particles, elaborated by Dirac,[1] was nonlocal and involved the introduction of nonphysical dynamical variables associated with strings. More recent formulations,[2–4] following the original work of Schwinger,[2] avoid string variables. Instead, they are based upon a Hamiltonian density which is a nonlocal function of the field variables, and an independently posited nonlocal commutation relation, which together yield nonlocal field equations.[5] In the present work the familiar apparatus of Lagrangian field theory is applied to yield directly the canonical commutation relations, the energy-momentum tensor, Lorentz transformation laws, and the formal expression for the S matrix given by Feynman's rules.

The present treatment thus brings the theory of magnetic monopoles close to standard quantum field theory, but it retains peculiarities characteristic of the theory of monopoles. The canonical quantization procedure applied to the local Lagrangian density yields nonlocal commutation relations between the potentials because the momentum canonical to one potential is the spatial derivative of another potential. This is the natural way the "Dirac string" enters into the local Lagrangian theory. Manifest isotropy of space-time is lost because the Lagrangian density depends on a fixed spacelike four-vector. Isotropy is regained only for quantized values of the coupling constants $(e_ng_m-g_ng_m)$, where e_n and g_m are electric and magnetic charges, as an integrability condition of a Lie algebra.

The transformation law of the field variables under finite change of Lorentz frame is found for the first time. We leave for another occasion the calculation of the transformation law of Green's functions and scattering amplitudes. In the meantime we only have the nonrelativistic transformation law, which is not that of a scalar, and a conjecture concerning the relativistic transformation law.[6]

In Sec. II, two electromagnetic four-potentials A^μ and B^μ and a local Lagrangian density are introduced which yield Maxwell's equations. In Sec. III an action principle is introduced for classical relativistic point electric and magnetic charges. (This section may be omitted without interrupting the local development.) In Sec. IV the Lagrangian density for electrically and magnetically charged spinor fields is introduced and the energy-momentum tensor is calculated. In Sec. V a definite choice of gauge is made and the canonical equal-time commutation relations are calculated. In Sec. VI the law of transformation of field variables under infinitesimal change of Lorentz frame is found. The condition that it integrate to a representation of the finite Lorentz group yields the charge-quantization law, $(e_ng_m-g_ne_m)/4\pi=Z_{nm}$, where Z_{nm} is an integer. In Sec. VII the Feynman rules are found. The equivalence of the present treatment to previous Hamiltonian formalisms[2–4] has been demonstrated, but we omit the proof which presents no new features.

II. INTRODUCTION OF POTENTIALS

We wish to find a Lagrangian form for Maxwell's equations[7]

$$\partial_\mu F^{\mu\nu}=j_e^{\ \nu}, \quad \partial_\mu F^{d\mu\nu}=j_g^{\ \nu}, \tag{2.1}$$

in the presence of conserved electric and magnetic currents j_e and j_g, where

$$\partial\cdot j_e=\partial\cdot j_g=0\,. \tag{2.2}$$

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[2] J. Schwinger, Phys. Rev. **144**, 1087 (1966); **151**, 1048 (1966); **151**, 1055 (1966); **173**, 1536 (1968); Science **165**, 757 (1969); **166**, 690 (1969).

[3] T. M. Yan, Phys. Rev. **150**, 1349 (1966); **155**, 1423 (1967).

[4] D. Zwanziger, Phys. Rev. **176**, 1489 (1968).

[5] N. Cabbibo and E. Ferrari, Nuovo Cimento **23**, 1147 (1962), give an alternative formulation in terms of path-dependent field variables.

[6] D. Zwanziger, Phys. Rev. **176**, 1480 (1968).

[7] We use the notations $g^{\mu\nu}=\text{diag}(1,\,-1,\,-1,\,-1)$, and $F^{d\mu\nu}\equiv\frac{1}{2}\epsilon^{\mu\nu}{}_{\kappa\lambda}F^{\kappa\lambda}$, where $\epsilon^{\mu\nu\kappa\lambda}$ is the completely antisymmetric symbol with $\epsilon^{0123}=1$.

As usual, we will introduce potentials. For this purpose observe that a particular solution of the second of Maxwell's equations is[8] $F^d=(n\cdot\partial)^{-1}(n\wedge j_g)$, where n is an arbitrary fixed four-vector and $(n\cdot\partial)^{-1}$ is an integral operator with kernel $(n\cdot\partial)^{-1}(x-y)$ satisfying $n\cdot\partial(n\cdot\partial)^{-1}(x)=\delta^4(x)$. The general solution to the second equation may be written

$$F^d=(\partial\wedge A)^d+(n\cdot\partial)^{-1}(n\wedge j_g)\,, \tag{2.3a}$$

$$F=(\partial\wedge A)-(n\cdot\partial)^{-1}(n\wedge j_g)^d\,, \tag{2.3b}$$

where A^μ is a four-potential which depends on the choice of gauge, the choice of n, and the determination of $(n\cdot\partial)^{-1}$. Similarly, the general solution to the first equation is

$$F=-(\partial\wedge B)^d+(n\cdot\partial)^{-1}(n\wedge j_e)\,, \tag{2.4a}$$

$$F^d=(\partial\wedge B)+(n\cdot\partial)^{-1}(n\wedge j_e)^d\,, \tag{2.4b}$$

where B^μ is another four-potential. It will be convenient to choose n spacelike so that these equations express $F(x)$ in terms of potentials at x and currents at points spacelike to x.

We may express F locally in terms of the potentials alone. Observe that any antisymmetric tensor G satisfies the identity[8]

$$G=(1/n^2)\{[n\wedge(n\cdot G)]-[n\wedge(n\cdot G^d)]^d\}\,, \tag{2.5}$$

since both left- and right-hand sides give the same value for $n\cdot G$ and $n\cdot G^d$. From Eqs. (2.3b) and (2.4b), we have

$$n\cdot F=n\cdot(\partial\wedge A)\,,\quad n\cdot F^d=n\cdot(\partial\wedge B)\,, \tag{2.6}$$

which yields the desired expression for F:

$$F=(1/n^2)(\{n\wedge[n\cdot(\partial\wedge A)]\}-\{n\wedge[n\cdot(\partial\wedge B)]\}^d)\,, \tag{2.7a}$$

$$F^d=(1/n^2)(\{n\wedge[n\cdot(\partial\wedge A)]\}^d+\{n\wedge[n\cdot(\partial\wedge B)]\})\,. \tag{2.7b}$$

We substitute these equations into Eq. (2.1) to obtain Maxwell's equations in terms of the potentials:

$$(1/n^2)(n\cdot\partial n\cdot\partial A^\mu-n\cdot\partial\,\partial^\mu n\cdot A-n^\mu n\cdot\partial\,\partial\cdot A+n^\mu\partial^2 n\cdot A-n\cdot\partial\,\epsilon^\mu{}_{\nu\kappa\lambda}n^\nu\partial^\kappa B^\lambda)=j_e{}^\mu\,, \tag{2.8a}$$

$$(1/n^2)(n\cdot\partial n\cdot\partial B^\mu-n\cdot\partial\,\partial^\mu n\cdot B-n^\mu n\cdot\partial\,\partial B+n^\mu\partial^2 n\cdot B+n\cdot\partial\,\epsilon^\mu{}_{\nu\kappa\lambda}n^\nu\partial^\kappa A^\lambda)=j_g{}^\mu\,. \tag{2.8b}$$

We have shown that corresponding to any solution F to Maxwell's equations there exist potentials A and B related to F by (2.7) and satisfying (2.8). Conversely, every pair of potentials satisfying (2.8) defines a unique solution F to Maxwell's equations given by (2.7). Of course A and B are highly non-unique, and we defer until we discuss canonical variables how to make them unique by appropriate choice of gauge and boundary conditions.

The equations of motion (2.8) follow from the Lagrangian density

$$\mathcal{L}=\mathcal{L}_\gamma+\mathcal{L}_I\,, \tag{2.9}$$

where

$$\mathcal{L}_\gamma=-(1/2n^2)[n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)^d]+(1/2n^2)[n\cdot(\partial\wedge B)]\cdot[n\cdot(\partial\wedge A)^d]-(1/2n^2)[n\cdot(\partial\wedge A)]^2-(1/2n^2)[n\cdot(\partial\wedge B)]^2 \tag{2.10}$$

and

$$\mathcal{L}_I=-j_e\cdot A-j_g\cdot B\,, \tag{2.11}$$

if the sources j_e and j_g are assumed given. The Lagrangian density may be written in a slightly different form which will be convenient later. From identity (2.5) we have the further identity

$$\mathrm{tr}(G\cdot G)=G_{\mu\nu}G^{\nu\mu}=(2/n^2)[-(n\cdot G)^2+(n\cdot G^d)^2]\,. \tag{2.12}$$

Letting $G=\partial\wedge A$ and $G=\partial\wedge B$, we find from (2.10)

$$\mathcal{L}_\gamma=\tfrac{1}{8}\,\mathrm{tr}[(\partial\wedge A)\cdot(\partial\wedge A)]+\tfrac{1}{8}\,\mathrm{tr}[(\partial\wedge B)\cdot(\partial\wedge B)]-\frac{1}{4n^2}\{n\cdot[(\partial\wedge A)+(\partial\wedge B)^d]\}^2-\frac{1}{4n^2}\{n\cdot[(\partial\wedge B)-(\partial\wedge A)^d]\}^2\,. \tag{2.13}$$

III. ACTION PRINCIPLE FOR CLASSICAL RELATIVISTIC ELECTRIC AND MAGNETIC CHARGES

The equations of motion of the quantum field theory will be formulated in terms of the potentials. Because they do not have a simple transformation law the problem of Lorentz invariance becomes acute and its correct resolution leads to charge quantization. Therefore, we prefer to gain insight by applying the Lagrangian formulation to classical relativistic point particles[9,10] where the equations of motion are manifestly covariant and the introduction of potentials is a luxury.

Maxwell's equations (2.1) are completed by specification of the currents

$$j_e{}^\mu(x)=\sum_i e_i\int\delta^4(x-x_i)u_i{}^\mu d\tau_i\,, \tag{3.1a}$$

$$j_g{}^\mu(x)=\sum_i g_i\int\delta^4(x-x_i)u_i{}^\mu d\tau_i\,, \tag{3.1b}$$

[8] It is convenient to suppress indices and write $(a\cdot G)^\nu\equiv a_\mu G^{\mu\nu}=-G^{\nu\mu}a_\mu\equiv-(G\cdot a)^\nu$, and also $(a\wedge b)^{\mu\nu}\equiv a^\mu b^\nu-a^\nu b^\mu$, for four-vectors a and b and antisymmetric tensors $G^{\mu\nu}$. Thus, for example, $G^{dd}=-G$; $a\cdot(b\wedge c)=a\cdot bc-a\cdot cb$; and $a\cdot(b\wedge c)^d$ is the four-vector $a_\mu\epsilon^{\mu\nu}{}_{\kappa\lambda}b^\kappa c^\lambda$.

[9] Within a somewhat different framework, F. Rohrlich, Phys. Rev. **150**, 1104 (1966), has in fact claimed that a Lagrangian does not exist for classical relativistic electric and magnetic charges.

[10] E. H. Kerner, J. Math. Phys. **11**, 39 (1970), has recently constructed a manifestly Galilean-invariant Lagrangian for classical nonrelativistic electrically and magnetically charged particles interacting instantaneously.

where the particle trajectories are specified by $x_i^\mu = x_i^\mu(\tau_i)$, $u_i^\mu = \dot{x}_i^\mu = dx_i^\mu/d\tau_i$, and τ_i parametrizes distance along the trajectory of the ith particle. The trajectories are determined by the Lorentz force

$$\frac{d}{d\tau_i}\left[\frac{m_i u_i}{(u_i^2)^{1/2}}\right] = [e_i F(x_i) + g_i F^d(x_i)]\cdot u_i. \quad (3.2)$$

When expressions (2.3b) and (2.4b) are used for F and F^d, the Lorentz force becomes

$$\frac{d}{d\tau_i}\left[\frac{m_i u_i}{(u_i^2)^{1/2}}\right] = \{e_i[\partial \wedge A(x_i)] + g_i[\partial \wedge B(x_i)]\}\cdot u_i - \sum_j (e_i g_j - g_i e_j) n \cdot \int (n\cdot\partial)^{-1} \times (x_i - x_j)(u_i \wedge u_j)^d d\tau_j, \quad (3.3)$$

and we see that the Lorentz force is made up of a term which is the expected local interaction with the potential plus an interparticle action at a distance depending on the kernel $(n\cdot\partial)^{-1}(x_i - x_j)$.

We restrict the determination of $(n\cdot\partial)^{-1}(x)$ to the form

$$(n\cdot\partial)^{-1}(x) = a\int_0^\infty \delta^4(x - ns)ds - (1-a)\int_0^\infty \delta^4(x + ns)ds, \quad (3.4)$$

so the support of $(n\cdot\partial)^{-1}(x_i - x_j)$ is reduced to

$$x_i^\mu(\tau_i) - x_j^\mu(\tau_j) = n^\mu s; \quad -\infty < s, \tau_i, \tau_j < \infty. \quad (3.5)$$

In general, this condition will not be satisfied anywhere along a trajectory unless it is exceptional because there are four equations but only three free parameters. Hence, in the general case the last term of Eq. (3.3) may be dropped, and we obtain[11]

$$\frac{d}{d\tau_i}\left[\frac{m_i u_i}{(u_i^2)^{1/2}}\right] = \{e_i[\partial \wedge A(x_i)] + g_i[\partial \wedge B(x_i)]\}\cdot u_i. \quad (3.6)$$

This equation of motion for the particle trajectories and the equations of motion of the potentials (2.8) result from requiring that the action

$$S = S_p + S_\gamma + S_I \quad (3.7)$$

be an extremum with respect to variation of $x_i^\mu(\tau_i)$ and $A(x)$ and $B(x)$, where

$$S_p = -\sum_i \int m_i (u_i^2)^{1/2} d\tau_i, \quad (3.8)$$

$$S_\gamma = \int \mathcal{L}_\gamma(x) d^4x, \quad (3.9)$$

$$S_I = \int \mathcal{L}_I(x) d^4x = -\int (j_e\cdot A + j_g \cdot B) d^4x = -\sum_i \int [e_i A(x_i) + g_i B(x_i)]\cdot u_i d\tau_i, \quad (3.10)$$

and $\mathcal{L}_\gamma$ is given by Eq. (2.10).

Before going on to the quantum-field-theory case, let us observe two peculiarities of the Lagrangian method presented here. First, as we have mentioned, the Lagrangian equations are defective for exceptional points of exceptional trajectories. Secondly, we derived our action principle by using Eqs. (2.3b) and (2.4b) for F and F^d with $(n\cdot\partial)^{-1}(x)$ given by (3.4), but the Lagrangian equations of motion only imply Eqs. (2.3b) and (2.4b) multiplied by $(n\cdot\partial)$. To see this, take the dual of the antisymmetric product of (2.8b) with n:

$$\frac{1}{n^2}(n\cdot\partial)\{(n\wedge[n\cdot(\partial\wedge B)])^d - (n\wedge[n\cdot(\partial\wedge A)^d])^d\} = (n\wedge j_g)^d.$$

Using (2.5) with $G = \partial \wedge A$, we obtain

$$n\cdot\partial\frac{1}{n^2}(\{n\wedge[n\cdot(\partial\wedge A)]\} - \{n\wedge[n\cdot(\partial\wedge B)]\}^d) = n\cdot\partial(\partial\wedge A) - (n\wedge j_g)^d,$$

and similarly

$$n\cdot\partial\frac{1}{n^2}(\{n\wedge[n\cdot(\partial\wedge B)]\} + \{n\wedge[n\cdot(\partial\wedge A)]\}^d) = n\cdot\partial(\partial\wedge B) + (n\wedge j_e)^d.$$

The left-hand sides are $(n\cdot\partial)$ times the expression for the fields in terms of the solutions A and B of the Lagrangian equations of motion. Hence, to ensure that the Lagrangian equations of motion for the particles agree with the Lorentz force, we must impose as boundary conditions on the solutions that they satisfy Eqs. (2.3) and (2.4) with $(n\cdot\partial)$ given by (3.4). Observe that by equating Eqs. (2.3b) and (2.4a), it follows that admissible solutions A and B satisfy

$$\partial\wedge A + (\partial\wedge B)^d = (n\cdot\partial)^{-1}[(n\wedge j_e) + (n\wedge j_g)^d]. \quad (3.11)$$

Conversely, one may show that this equation implies both the Lagrangian equations (2.8) and also the boundary conditions (2.3) and (2.4), when F is defined by (2.7). Thus, it has all the desired properties of an equation of motion for the potentials.

We will see later how each of the peculiarities of the Lagrangian theory recurs in the quantum field theory. The boundary conditions will be imposed as boundary

[11] At exceptional points of exceptional trajectories where condition (3.5) holds, the right-hand side of Eq. (3.6) in fact becomes singular and it should be solved by continuity.

conditions on equal-time commutators between the potentials, while incorporation of the exceptional configurations restricts the charges to quantized values.

IV. SPINOR FIELDS AND ENERGY-MOMENTUM TENSOR

Let ψ_n be a set of Dirac fields each with electric and magnetic charges e_n and g_n, so the currents are[7,12]

$$j_e^\mu=\sum_n e_n\bar{\psi}_n\gamma^\mu\psi_n\,,\quad j_g^\mu=\sum_n g_n\bar{\psi}_n\gamma^\mu\psi_n\,. \tag{4.1}$$

By correspondence with the classical action (3.7), we obtain the action function for the spinor fields:

$$S=\int \mathcal{L}(x)d^4x\,,\quad \mathcal{L}=\mathcal{L}_\gamma+\mathcal{L}_M+\mathcal{L}_I\,, \tag{4.2}$$

with $\mathcal{L}_\gamma$ given by Eqs. (2.10) or (2.13) and

$$\mathcal{L}_M=\sum_n \bar{\psi}_n(i\gamma\cdot\partial-m)\psi_n\,, \tag{4.3}$$

$$\mathcal{L}_I=-(j_e\cdot A+j_g\cdot B)=-\sum_n \bar{\psi}_n\gamma\cdot(e_nA+g_nB)\psi_n\,. \tag{4.4}$$

The Lagrangian equations of motion for the potentials are (2.8), with the currents given by Eq. (4.1). For the Dirac fields, we have

$$[\gamma\cdot(i\partial-e_nA-g_nB)-m_n]\psi_n=0\,. \tag{4.5}$$

Because of the appearance of the fixed four-vector n in $\mathcal{L}_\gamma$, the crucial question is Lorentz-transformation properties, which we proceed to elucidate. Let ϕ_α be the set of fields A, B, ψ_n, so the Lagrangian depends locally on these fields and on the fixed four-vector n,

$$\mathcal{L}(x)=\mathcal{L}(\phi_\alpha(x),\partial_\alpha\phi_\alpha(x),n) \tag{4.6}$$

and all the equations of motion take the form

$$\frac{\partial\mathcal{L}}{\partial\phi_\alpha}-\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_\alpha}=0\,. \tag{4.7}$$

Corresponding to a change in the fields $\delta\phi_\alpha$, the Lagrangian changes by

$$\delta\mathcal{L}=\partial_\mu\sum_\alpha\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_\alpha}\delta\phi_\alpha\right) \tag{4.8}$$

as one verifies using the equations of motion. For a^ν, an infinitesimal displacement, we have in the standard way $\delta\mathcal{L}=a\cdot\partial\mathcal{L}$, $\delta\phi=a\cdot\partial\phi$, so

$$a\cdot\partial\mathcal{L}=\partial_\mu\sum_\alpha\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_\alpha}a\cdot\partial\phi_\alpha\right).$$

[12] We use the convention $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}$, $\bar{\psi}=\psi^\dagger\gamma^0$.

Since a^ν is arbitrary, we obtain

$$\partial_\mu T^{\mu\nu}=0\,, \tag{4.9}$$

with

$$T^{\mu\nu}=\sum_\alpha\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_\alpha}\partial^\nu\phi_\alpha-g^{\mu\nu}\mathcal{L}\,, \tag{4.10}$$

the conserved nonsymmetric energy-momentum tensor.

Lorentz transformations present new features. For any Lorentz transformation Λ, the Lagrangian (4.6) satisfies the identity

$$\mathcal{L}(\Lambda x)=\mathcal{L}[\phi_\alpha(\Lambda x),\Lambda_\mu{}^\nu\partial_\nu\phi_\alpha(\Lambda x),n]\,,$$
$$\mathcal{L}(\Lambda x)=\mathcal{L}[D_\alpha^{-1}(\Lambda)\phi_\alpha(\Lambda x),\;\partial_\mu D_\alpha^{-1}(\Lambda)\phi_\alpha(\Lambda x),\Lambda^{-1}n]\,, \tag{4.11}$$

where $D_\alpha(\Lambda)$ is given by Λ for $\phi_\alpha=A$ or B and by $S(\Lambda)$ for $\phi_\alpha=\psi_n$, with $S(\Lambda)$ the usual transformation matrix for a Dirac spinor. Let the Lorentz transformation Λ be infinitesimal, $\Lambda_\mu{}^\nu=g_\mu{}^\nu+\Omega_\mu{}^\nu$, where $\Omega_{\mu\nu}$ is an infinitesimal antisymmetric matrix, so

$$\mathcal{L}(\Lambda x)=\mathcal{L}(x+\Omega x)=\mathcal{L}(x)-x\cdot\Omega\cdot\partial\mathcal{L}(x)=\mathcal{L}(x)+\delta\mathcal{L}(x)\,.$$

The corresponding changes in ϕ_α are

$$\delta\phi_\alpha=-\tfrac{1}{2}\Omega_{\mu\nu}m_\alpha{}^{\mu\nu}\phi_\alpha=-(x\cdot\Omega\cdot\partial)\phi_\alpha-\tfrac{1}{2}\Omega_{\mu\nu}\Sigma_\alpha{}^{\mu\nu}\phi_\alpha\,, \tag{4.12}$$

where the antisymmetric matrices $\Sigma_\alpha{}^{\mu\nu}$ are defined by

$$\tfrac{1}{2}\Omega_{\mu\nu}\Sigma^{\mu\nu}A_\kappa=\Omega_\kappa{}^\lambda A_\lambda\,,\quad(\text{and }A\to B) \tag{4.13}$$

$$\tfrac{1}{2}\Omega_{\mu\nu}\Sigma^{\mu\nu}\psi_n=\tfrac{1}{4}\gamma\cdot\Omega\cdot\gamma\psi_n\,. \tag{4.14}$$

In terms of infinitesimals, (4.11) becomes

$$-x\cdot\Omega\cdot\partial\mathcal{L}=-\partial_\kappa\sum_\alpha\frac{\partial\mathcal{L}}{\partial\partial_\kappa\phi_\alpha}(\tfrac{1}{2}\Omega_{\mu\nu}m_\alpha{}^{\mu\nu}\phi_\alpha)+n\cdot\Omega\cdot\frac{\partial}{\partial n}\mathcal{L}\,. \tag{4.15}$$

Defining $M^{\kappa\mu\nu}$ by

$$M^{\kappa\mu\nu}=\sum_\alpha\frac{\partial}{\partial\partial_\kappa\phi_\alpha}m_\alpha{}^{\mu\nu}\phi_\alpha-(x^\mu g^{\nu\kappa}-x^\nu g^{\mu\kappa})\mathcal{L}\,, \tag{4.16}$$

or, by Eqs. (4.10) and (4.12),

$$M^{\kappa\mu\nu}=x^\mu T^{\kappa\nu}-x^\nu T^{\kappa\mu}+\sum_\alpha\frac{\partial\mathcal{L}}{\partial\partial_\kappa\phi_\alpha}\Sigma_\alpha{}^{\mu\nu}\phi_\alpha\,, \tag{4.17}$$

we have from Eq. (4.15), since Ω is arbitrary,

$$\partial_\kappa M^{\kappa\mu\nu}=(n^\mu\partial_n{}^\nu-n^\nu\partial_n{}^\mu)\mathcal{L}\,. \tag{4.18}$$

All of this is standard except for the nonvanishing divergence of $M^{\kappa\mu\nu}$ which represents the nonconservation of (four-dimensional) angular momentum due to the presence of the fixed vector n in $\mathcal{L}$. Let us evaluate the violating term $n\wedge\partial_n\mathcal{L}$. The four-vector n only appears in $\mathcal{L}_\gamma$, so we have from Eq. (2.13)

$$n\wedge\partial_n\mathcal{L}$$

$$=\frac{1}{2n^2}n\wedge\{n\cdot[\partial\wedge A+(\partial\wedge B)^d]\cdot[\partial\wedge A+(\partial\wedge B)^d]\}$$

$$+\frac{1}{2n^2}n\wedge\{n\cdot[\partial\wedge B-(\partial\wedge A)^d]\cdot[\partial\wedge B-(\partial\wedge A)^d]\}.$$

Making use of Eq. (3.11), we easily obtain

$$n\wedge\partial_n\mathcal{L}=-n\wedge\{n\cdot[(n\cdot\partial)^{-1}j_e\wedge(n\cdot\partial)^{-1}j_g]^d\}\,. \quad (4.19)$$

Let us see under what conditions it vanishes. First of all, we note that it depends on the coupling constants e_n and g_n only through the combination

$$\mu_{mn}=(e_m g_n-g_m e_n)\,.$$

Hence it vanishes if all electric and magnetic charges are proportional, i.e., $g_n=ce_n$ (all n) and in particular if $g_n=0$ (all n). This result is nontrivial for it shows that angular momentum may be conserved even though the Lagrangian itself is not rotationally symmetric, and this includes the case of ordinary electrodynamics. Secondly, we observe that because of the support properties of $(n\cdot\partial)^{-1}(x)$, expression (4.19) vanishes for classical currents everywhere except for the exceptional points of exceptional trajectories where, as we have seen, the Lagrangian equations of motion are not correct. To see this we note that (4.19) vanishes unless both $x=x_i(\tau_i)+ns_1$ and $x=x_j(\tau_j)+ns_2$ or $x_i(\tau_i)-x_j(\tau_j)=n(s_2-s_1)$, which is the condition for the exceptional configurations. Hence we may say that in the classical case $M^{\kappa\mu\nu}$ is a conserved angular momentum current. We defer until after quantization the discussion of spinor currents with $e_m g_n-g_m e_n\neq 0$.

Inserting Eq. (4.17) into Eq. (4.18) and using (4.9) and (4.19), we obtain

$$T^{\mu\nu}-T^{\nu\mu}+\partial_\kappa\sum_\alpha\frac{\partial}{\partial\partial_\kappa\phi_\alpha}\Sigma_\alpha{}^{\mu\nu}\phi_\alpha$$

$$=-(n\wedge\{n\cdot[(n\cdot\partial)^{-1}j_e\wedge(n\cdot\partial)^{-1}j_g]^d\})^{\mu\nu}.$$

Because of the nonvanishing right-hand side, we cannot completely symmetrize $T^{\mu\nu}$, but proceeding as usual, we define

$$\theta^{\mu\nu}=T^{\mu\nu}+\tfrac{1}{2}\partial_\kappa\sum_\alpha\left[\frac{\partial}{\partial\partial_\kappa\phi_\alpha}\Sigma_\alpha{}^{\mu\nu}\phi_\alpha-\frac{\partial}{\partial\partial_\mu\phi_\alpha}\Sigma_\alpha{}^{\kappa\nu}\phi_\alpha-\frac{\partial}{\partial\partial_\nu\phi_\alpha}\Sigma_\alpha{}^{\kappa\mu}\phi_\alpha\right]$$

or

$$\theta^{\mu\nu}=\tfrac{1}{2}(T^{\mu\nu}+T^{\nu\mu})-\tfrac{1}{2}\partial_\kappa\sum_\alpha\left(\frac{\partial}{\partial\partial_\mu\phi_\alpha}\Sigma_\alpha{}^{\kappa\nu}\phi_\alpha+\frac{\partial}{\partial\partial_\nu\phi_\alpha}\Sigma_\alpha{}^{\kappa\mu}\phi_\alpha\right)$$

$$-(n\wedge n\cdot\{n\cdot[(n\cdot\partial)^{-1}j_e\wedge(n\cdot\partial)^{-1}j_g]^d\})^{\mu\nu}, \quad (4.20)$$

which is almost completely symmetric. It is differentially conserved,

$$\partial_\mu\theta^{\mu\nu}=0\,, \quad (4.21)$$

and its spatial integral

$$P^\nu=\int\theta^{0\nu}d^3x=\int T^{0\nu}d^3x \quad (4.22)$$

is the conserved energy-momentum vector. In evaluating (4.20) explicitly, considerable simplification results if one uses Eqs. (2.3), (2.4), (3.11), and the identity

$$(-H\cdot G+G^d\cdot H^d)^{\mu\nu}=-\tfrac{1}{2}g^{\mu\nu}\operatorname{tr}(G\cdot H)\,, \quad (4.23)$$

where G and H are antisymmetric tensors. One obtains

$$\theta^{\mu\nu}=\tfrac{1}{2}(F\cdot F+F^d\cdot F^d)^{\mu\nu}$$

$$+\tfrac{1}{2}\sum_n\bar\psi_n[\gamma^\mu(i\partial-e_n\cdot A-g_n\cdot B)^\nu+\gamma^\nu(i\partial-e_n\cdot A-g_n\cdot B)^\mu]\psi_n$$

$$-n^\mu\{n\cdot[(n\cdot\partial)^{-1}j_e\wedge(n\cdot\partial)^{-1}j_g]^d\}^\nu, \quad (4.24)$$

where ∂ is half the derivative to the right less the derivative to the left. The first part is just what we expect for charged Dirac particles interacting with the electromagnetic field. The last term is spurious because, as we have seen, in the classical case it is nonvanishing only for exceptional points of exceptional trajectories, where the Lagrangian method itself is not correct. In any case, it does not contribute to the integrals P^ν, Eq. (4.22), if the time axis is chosen perpendicular to n^μ. In such a frame, it does not contribute either to

$$M^{\mu\nu}\equiv\int(x^\mu\theta^{0\nu}-x^\nu\theta^{0\mu})d^3x. \quad (4.25)$$

For the time derivative of $M^{\mu\nu}$, we find

$$\frac{d}{dt}M^{\mu\nu}=\int(\theta^{\mu\nu}-\theta^{\nu\mu})d^3x$$

$$=-\int(n\wedge\{n\cdot[(n\cdot\partial)^{-1}j_e\wedge(n\cdot\partial)^{-1}j_g]^d\})^{\mu\nu}d^3x. \quad (4.26)$$

Choosing the three-axis along n^μ, we see that M^{03}, M^{13}, and M^{23} are violated by the spurious term, and that the other seven generators of the Poincaré group are manifestly conserved.

V. CANONICAL QUANTIZATION

In the usual formulation of quantum electrodynamics, A satisfies a second-order differential equation and B does not appear. Here we have doubled the number of variables, so they should satisfy a first-order equation. If we examine the Lagrangian equations of motion (2.8) we observe that except for the terms $\partial^2 n\cdot A$ and $\partial^2 n\cdot B$, the second-order differential operator which appears there factorizes into $n\cdot\partial$ and another first-order differential operator. If the factorization were perfect, we would have, in effect, a first-order system. The unwanted terms could be eliminated by imposing the axial gauge conditions,[13]

$$n\cdot A=n\cdot B=0\,. \tag{5.1}$$

All equations up to now have been invariant under the substitutions

$$A_\mu\rightarrow A_\mu+\partial_\mu\chi_e, \tag{5.2a}$$

$$B_\mu\rightarrow B_\mu+\partial_\mu\chi_g, \tag{5.2b}$$

$$\psi_n\rightarrow \exp[-ie_n\chi_e-ig_n\chi_g]\psi_n, \tag{5.2c}$$

where χ_e and χ_g are a pair of arbitrary functions of spacetime, and a choice of gauge such as (5.1) is in fact necessary to make A and B well defined. The constraint (5.1) on the equation of motion (2.8) leads to another constraint. Contract Eqs. (2.8) with a timelike vector τ orthogonal to n:

$$(1/n^2)n\cdot\partial[n\cdot\partial\tau\cdot A-\tau\cdot(\partial\wedge B)^d\cdot n]-\tau\cdot j_e=0, \tag{5.3a}$$

$$(1/n^2)n\cdot\partial[n\cdot\partial\tau\cdot B+\tau\cdot(\partial\wedge A)^d\cdot n]-\tau\cdot j_g=0\,. \tag{5.3b}$$

These equations constitute another pair of constraints because they contain derivatives only along directions perpendicular to τ. Equations (5.1) and (5.3) reduce the four degrees of A and B to the two degrees of freedom appropriate to a massless spin-one field. However, incorporation of these constraints into a canonical formalism is clumsy, though feasible, and we prefer a slightly different but entirely equivalent procedure.

The method will be to add a gauge-dependent term $\mathcal{L}_G$ to the Lagrangian, which will ensure that the free-field equations

$$\partial^2 n\cdot A+\partial^2 n\cdot B=0 \tag{5.4}$$

are satisfied as equations of motion to replace the constraints $n\cdot A=n\cdot B=0$. This is enough to eliminate the unwanted terms from Eqs. (2.8). In fact, if we put

$$\mathcal{L}_G=(1/2n^2)\{[\partial(n\cdot A)]^2+[\partial(n\cdot B)]^2\}$$

and take as total Lagrangian density

$$\mathcal{L}=\mathcal{L}_{em}+\mathcal{L}_M+\mathcal{L}_I \tag{5.5}$$

with $\mathcal{L}_M$ and $\mathcal{L}_I$ defined by Eqs. (4.3) and (4.4) and $\mathcal{L}_{em}$ by

$$\begin{aligned}\mathcal{L}_{em}=\mathcal{L}_\gamma+\mathcal{L}_G=&-(1/2n^2)[n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)^d]\\&+(1/2n^2)[n\cdot(\partial\wedge B)]\cdot[n\cdot(\partial\wedge A)^d]\\&-(1/2n^2)[(n\cdot\partial A)^2-2(n\cdot\partial A)\cdot(\partial n\cdot A)\\&\qquad+(n\cdot\partial B)^2-2(n\cdot\partial B)\cdot(\partial n\cdot B)],\end{aligned} \tag{5.6}$$

then the Lagrangian equations of motion for the potentials take the desired factorized form

$$\frac{n\cdot\partial}{n^2}(n\cdot\partial A^\mu-\partial^\mu n\cdot A-n^\mu\partial\cdot A-\epsilon^\mu{}_{\nu\kappa\lambda}n^\nu\partial^\kappa B^\lambda)=j_e{}^\mu, \tag{5.7a}$$

$$\frac{n\cdot\partial}{n^2}(n\cdot\partial B^\mu-\partial^\mu n\cdot B-n^\mu\partial\cdot B+\epsilon^\mu{}_{\nu\kappa\lambda}n^\nu\partial^\kappa A^\lambda)=j_g{}^\mu. \tag{5.7b}$$

These are the old equations of motion (2.8), with the $\partial^2 n\cdot A$ and $\partial^2 n\cdot B$ term missing. We impose as boundary conditions that the potentials satisfy the corresponding integral equations

$$(1/n^2)(n\cdot\partial A^\mu-\partial^\mu n\cdot A-n^\mu\partial\cdot A-\epsilon^\mu{}_{\nu\kappa\lambda}n^\nu\partial^\kappa B^\lambda)=(n\cdot\partial)^{-1}j_{e\mu}, \tag{5.8a}$$

$$(1/n^2)(n\cdot\partial B^\mu-\partial^\mu n\cdot B-n^\mu\partial\cdot B+\epsilon^\mu{}_{\nu\kappa\lambda}n^\nu\partial^\kappa A^\lambda)-(n\cdot\partial)^{-1}j_g{}^\mu, \tag{5.8b}$$

with $(n\cdot\partial)^{-1}$ given by Eq. (3.4). On taking the divergence of these equations and using current conservation, one recovers the free-field equations (5.4) which together with Eqs. (5.7) yield the old equations of motion (2.8). On contracting Eqs. (5.8) with n, we obtain the gauge conditions

$$\partial\cdot A=-(n\cdot\partial)^{-1}n\cdot j_e, \tag{5.9a}$$

$$\partial\cdot B=-(n\cdot\partial)^{-1}n\cdot j_g, \tag{5.9b}$$

which restrict the gauge functions χ_e and χ_g of Eqs. (5.2) to solutions of the free wave equation. If one takes the antisymmetric product of Eqs. (5.8) with n, and uses identity (2.5), then the integral equation (3.11) results. Finally, if we contract Eqs. (5.8) with a timelike four-vector τ orthogonal to n, we obtain

$$\tau\cdot\partial n\cdot A=n\cdot\partial\tau\cdot A-\tau\cdot(\partial\wedge B)^d\cdot n-n^2(n\cdot\partial)^{-1}\tau\cdot j_e, \tag{5.10a}$$

$$\tau\cdot\partial n\cdot B=n\cdot\partial\tau\cdot B+\tau\cdot(\partial\wedge A)^d\cdot n-n^2(n\cdot\partial)^{-1}\tau\cdot j_g\,. \tag{5.10b}$$

These equations state that also quantities (5.3), which in the axial gauge are constrained to zero, are instead free fields because they are time derivatives of the free fields $n\cdot A$ and $n\cdot B$.

The additional gauge-dependent Lagrangian $\mathcal{L}_G$ adds an extra term $T_G{}^{\mu\nu}$ to the $T^{\mu\nu}$ tensor (4.10):

$$T_G{}^{\mu\nu}=(1/n^2)[\partial^\mu(n\cdot A)\partial^\nu(n\cdot A)+\partial^\mu(n\cdot B)\partial^\nu(n\cdot B)]-g^{\mu\nu}\mathcal{L}_G\,. \tag{5.11}$$

It is symmetric and adds unchanged[14] to $\theta^{\mu\nu}$, Eq. (4.24),

[13] B. Zumino, in *International School of Physics, Ettore Majorana*, edited by A. Zichichi (Academic, New York, 1966), advocates the axial gauge for magnetic monopoles.

[14] If one carries out systematically the procedure of Sec. IV with the extra term $\mathcal{L}_G$, Eq. (4.16) gets an additional contribution on the right:

$$[n\wedge\partial_n]^{\mu\nu}\mathcal{L}_G=\partial_\kappa\{(1/n^2)[(n\wedge A)^{\mu\nu}\partial^\kappa n\cdot A+(n\wedge B)^{\mu\nu}\partial^\kappa n\cdot B]\}.$$

This does not vanish, but because it is a four-divergence, it may be brought to the left-hand side. The final result is Eq. (5.12).

so we have,

$$\theta^{\mu\nu}=\tfrac{1}{2}(F\cdot F+F^d\cdot F^d)^{\mu\nu}$$
$$+\tfrac{1}{2}\sum_n \bar\psi_n[\gamma^\mu(i\partial^\nu-e_nA^\nu-g_nB^\nu)$$
$$+\gamma^\nu(i\partial^\mu-e_nA^\mu-g_nB^\mu)]\psi_n$$
$$+(1/n^2)[\partial^\mu n\cdot A\,\partial^\nu n\cdot A+\partial^\mu n\cdot B\,\partial^\nu n\cdot B$$
$$-g^{\mu\nu}\tfrac{1}{2}(\partial n\cdot A)^2+(\partial n\cdot B)^2]$$
$$-n^\mu\{n\cdot[(n\cdot\partial)^{-1}j_e\wedge(n\cdot\partial)^{-1}j_g]^d\}^\nu. \quad (5.12)$$

We are now ready to consider the important question of the equal-time canonical commutation relations which correspond to the Lagrangian (5.5). We choose the time axis orthogonal to n,

$$n^\mu=(0,\mathbf{n}), \quad (5.13)$$

which is assumed to hold henceforth, so that the Lagrangian equations for the potentials (5.7) are of first order in the time derivatives. In order to find four pairs of canonically conjugate variables for this first-order system with eight independent variables A^μ and B^μ, we must rewrite the Lagrangian in the form

$$\mathcal{L}=\sum_{\alpha=1}^{4}\pi_\alpha(A,B)\dot\phi_\alpha(A,B)-\mathcal{H}(A,B)+\partial_\mu X^\mu, \quad (5.14)$$

where $\partial_\mu X^\mu$ are exact derivatives and all other time derivatives appear explicitly (all dependence on the spinors is suppressed). Then $\phi_\alpha(A,B)$ and $\pi_\alpha(A,B)$ may be identified as canonical conjugate variables. Comparison with Eq. (5.6) shows that we may write

$$\sum_{\alpha=1}^{4}\pi_\alpha\dot\phi_\alpha=-n\cdot\partial B^2\dot A^1+n\cdot\partial A^2\dot B^1+n\cdot\partial A^0\dot A^3+n\cdot\partial B^0\dot B^3,$$

where we have temporarily taken n along the three-axis, $n^\mu=(0,0,0,1)$. Hence we make the identifications

$$\phi_1=A^1,\quad \phi_2=B^1,\quad \phi_3=A^3,\quad \phi_4=B^3,$$
$$B^2=-(n\cdot\partial)^{-1}\pi_1,\quad A^2=(n\cdot\partial)^{-1}\pi_2,$$
$$A^0=(n\cdot\partial)^{-1}\pi_3,\quad B^0=(n\cdot\partial)^{-1}\pi_4.$$

In the last relation, we choose the determination of $(n\cdot\partial)^{-1}$ provided by Eq. (3.4). This is a boundary condition which goes beyond the canonical identification and which ensures that not only the Lagrangian equation (5.7), but also the integral equation (5.8) will be a consequence of the Heisenberg equations of motion.

The equal-time canonical commutation relations

$$[\phi_\alpha(t,\mathbf{x}),\phi_\beta(t,\mathbf{y})]=[\pi_\alpha(t,\mathbf{x}),\pi_\beta(t,\mathbf{y})]=0,$$
$$[\pi_\alpha(t,\mathbf{x}),\phi_\beta(t,\mathbf{y})]=-i\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y}),$$

determine the desired nonvanishing equal-time commutation relations between the potentials:

$$[A^\mu(t,\mathbf{x}),B^\nu(t,\mathbf{y})]=i\epsilon^{\mu\nu}{}_{\kappa0}n^\kappa(n\cdot\partial)^{-1}(x-y), \quad (5.15a)$$
$$[A^\mu(t,\mathbf{x}),A^\nu(t,\mathbf{y})]=[B^\mu(t,\mathbf{x}),B^\nu(t,\mathbf{y})]$$
$$=-i(g_0{}^\mu n^\nu+g_0{}^\nu n^\mu)(n\cdot\partial)^{-1}(\mathbf{x}-\mathbf{y}), \quad (5.15b)$$

where we have restored n to the more general form (5.13). Here $(n\cdot\partial)^{-1}$ means the three-dimensional[15] inverse of $(n\cdot\partial)=(n^\mu\partial_\mu)=(\mathbf{n}\cdot\nabla)$, so from Eq. (3.4)

$$(\mathbf{n}\cdot\nabla)^{-1}(\mathbf{x})$$
$$=a\int_0^\infty\delta^3(\mathbf{x}-\mathbf{n}s)ds-(1-a)\int_0^\infty\delta^3(\mathbf{x}+\mathbf{n}s)ds. \quad (5.16)$$

The equal-time commutation relations between the potentials (5.15) are the principal result of this section. They have been obtained as a consequence of the canonical quantization procedure.

The constant a of Eq. (5.16) is yet to be determined. All equations heretofore are invariant under the simultaneous substitutions,

$$(e_n,g_n)\to(\cos\theta\, e_n-\sin\theta\, g_n,\ \sin\theta\, e_n+\cos\theta\, g_n), \quad (5.17a)$$
$$(A,B)\to(\cos\theta\, A-\sin\theta\, B,\ \sin\theta\, A+\cos\theta\, B), \quad (5.17b)$$

which imply

$$(F,F^d)\to(\cos\theta\, F-\sin\theta\, F^d,\ \sin\theta\, F+\cos\theta\, F^d). \quad (5.17c)$$

This chiral invariance means that absolute directions in the two-dimensional charge space (e,g) are unobservable. If we also want the commutation relations (5.15) to be invariant under this substitution, then we must have $a=\frac{1}{2}$ and hence

$$(\mathbf{n}\cdot\nabla)^{-1}(\mathbf{x})=\int_{-\infty}^{\infty}\delta^3(\mathbf{x}-\mathbf{n}s)\tfrac{1}{2}\epsilon(s)ds. \quad (5.18)$$

This has been discussed in detail in Ref. 4.

The quantization of the ψ_n's is ordinary:

$$\{\psi_m(\mathbf{x}),\psi_n(\mathbf{y})\}=0;$$
$$\{\psi_m(\mathbf{x}),\psi_n{}^\dagger(\mathbf{y})\}=\delta(\mathbf{x}-\mathbf{y})\delta_{mn}, \quad (5.19)$$

and the ψ_n commute with A and B. From Eqs. (2.3) and (2.4) we express the electromagnetic fields in terms of the canonical variables[15]:

$$-F^{d0i}=-\tfrac{1}{2}\epsilon_{ijk}F^{jk}=\mathbf{H}=\nabla\times\mathbf{A}$$
$$+(\mathbf{n}\cdot\nabla)^{-1}\mathbf{n}\rho_g\equiv H_i, \quad (5.20a)$$
$$-F^{0i}=\tfrac{1}{2}\epsilon_{ijk}F^{djk}=\mathbf{E}=-\nabla\times\mathbf{B}$$
$$+(\mathbf{n}\cdot\nabla)^{-1}\mathbf{n}\rho_e\equiv E_i, \quad (5.20b)$$

which gives the standard equal-time commutation

[15] The relation of three- and four-vector notation is $x^\mu=(t,\mathbf{x})$, $j^\mu=(\rho,\mathbf{j})$, $A^\mu=(A^0,\mathbf{A})$, $B^\mu=(B^0,\mathbf{B})$, $n^\mu=(n^0,\mathbf{n})$, $\partial_\mu=(\partial_t,\nabla)$, $\gamma^\mu=(\gamma^0,\boldsymbol{\gamma})$, and ϵ_{ijk} is the three-dimensional antisymmetric symbol with $\epsilon_{123}=1$.

relation for the electromagnetic fields,

$$[E_i(\mathbf{x}),H_j(\mathbf{y})]=i\epsilon_{ijk}\nabla_k\delta(\mathbf{x}-\mathbf{y})\,,$$
$$[E_i(\mathbf{x}),E_j(\mathbf{y})]=[H_i(\mathbf{x}),H_j(\mathbf{y})]=0\,, \quad (5.21)$$

and the important commutator between the fields and ψ_n,

$$[\mathbf{E}(\mathbf{x}),\psi_n(\mathbf{y})]=-e_n\mathbf{n}(\mathbf{n}\cdot\nabla)^{-1}(x-y)\psi_n(\mathbf{y})\,, \quad (5.22a)$$

$$[\mathbf{H}(\mathbf{x}),\psi_n(\mathbf{y})]=-g_n\mathbf{n}(\mathbf{n}\cdot\nabla)^{-1}(\mathbf{x}-\mathbf{y})\psi_n(y)\,. \quad (5.22b)$$

The Hamiltonian density of Eq. (5.14) equals θ^{00} of Eq. (5.12):

$$\theta^{00}=\tfrac{1}{2}(\mathbf{E}^2+\mathbf{H}^2)+\sum_n \bar{\psi}_n\gamma^0(i\partial^0-e_nA^0-g_nB^0)\psi_n$$
$$-(1/2n^2)[\partial^0(\mathbf{n}\cdot\mathbf{A})^2+(\partial^0\mathbf{n}\cdot\mathbf{B})^2$$
$$+(\nabla\mathbf{n}\cdot\mathbf{A})^2+(\nabla\mathbf{n}\cdot\mathbf{B})^2]. \quad (5.23)$$

The last term, which arises from the gauge-dependent Lagrangian $\mathcal{L}_G$, contributes to θ^{00} the negative of the energy density of two free scalar fields, $(\mathbf{n}\cdot\mathbf{A})$ and $(\mathbf{n}\cdot\mathbf{B})$. Physical states may be restricted by the subsidiary condition that they be vacuum states with respect to these free fields.

We use Eqs. (5.20) and the equations of motion (4.5) and (5.10) to express θ^{00} in terms of the canonical variable A, B, and ψ_n,

$$\theta^{00}=\tfrac{1}{2}[(\nabla\times\mathbf{A})^2+(\nabla\times\mathbf{B})^2]$$
$$-(1/2\mathbf{n}^2)[(\mathbf{n}\cdot\nabla A^0-\mathbf{n}\cdot\nabla\times\mathbf{B})^2$$
$$+(\mathbf{n}\cdot\nabla B^0+\mathbf{n}\cdot\nabla\times\mathbf{A})^2+(\nabla\mathbf{n}\cdot\mathbf{A})^2+(\nabla\mathbf{n}\cdot\mathbf{B})^2]$$
$$+\sum_n \bar{\psi}_n(\boldsymbol{\gamma}\cdot\mathbf{p}+m)\psi_n+j_e\cdot A+j_e\cdot B$$
$$-\mathbf{n}\cdot\nabla[A^0(\mathbf{n}\cdot\nabla)^{-1}\rho_e+B^0(\mathbf{n}\cdot\nabla)^{-1}\rho_g]. \quad (5.24)$$

The last term is a divergence and does not contribute to the Hamiltonian. Thus, the interaction Hamiltonian $\int(j_e\cdot A+j_e\cdot B)d^3x$ is the negative of the interaction Lagrangian. One easily verifies that the equations of motion (4.5) and the integral equations (5.8) follow from the commutation relations (5.15) and (5.19) and the Heisenberg equations of motion

$$i\dot{\phi}_\alpha=[\phi_\alpha,H]\,, \quad (5.25)$$

where

$$H=\int\theta^{00}d^3x \quad \text{and} \quad \phi_\alpha \quad \text{is} \quad A^\mu,\, B^\mu,\, \text{or}\ \psi_n.$$

Finally, we note the expression for the momentum density in terms of canonical variables,

$$\theta^{0i}=(\nabla\times\mathbf{A})\times(\nabla\times\mathbf{B})$$
$$-(1/\mathbf{n}^2)[\mathbf{n}\cdot\nabla A^0-\mathbf{n}\cdot\nabla\times\mathbf{B}]\nabla\mathbf{n}\cdot\mathbf{A}$$
$$-(1/\mathbf{n}^2)[\mathbf{n}\cdot\nabla\mathbf{B}^0+\mathbf{n}\cdot\nabla\times\mathbf{A}]\nabla\mathbf{n}\cdot\mathbf{B}$$
$$+\sum_n\psi_n^\dagger(-i)\nabla\psi_n+\tfrac{1}{4}\nabla\times\sum_n\psi_n^\dagger\boldsymbol{\sigma}\psi_n$$
$$-\mathbf{n}\cdot\nabla[\mathbf{A}(n\cdot\nabla)^{-1}\rho_e+\mathbf{B}(n\cdot\nabla)^{-1}\rho_g], \quad (5.26)$$

where $\boldsymbol{\sigma}=+\tfrac{1}{2}i\boldsymbol{\gamma}\times\boldsymbol{\gamma}$. One easily verifies that the momentum operator $\mathbf{P}=\int\theta^{0i}d^3x$ satisfies

$$-i\nabla\phi_\alpha=[\phi_\alpha,\mathbf{P}]\,. \quad (5.27)$$

Equations (5.25) and (5.27) together have the covariant form

$$i\partial^\mu\phi_\alpha=[\phi_\alpha,P^\mu]\,. \quad (5.28)$$

VI. COVARIANCE AND CHARGE QUANTIZATION

As a guide to the field-theoretic case, we will first briefly review the nonrelativistic quantum-mechanical monopole problem from a point of view that lends itself to generalization. A more detailed discussion is provided by Hurst.[16] The nonrelativistic interaction of a pair of particles with charges (e_1,g_1) and (e_2,g_2) is defined, after elimination of center-of-mass motion, by the standard canonical commutation relations and the Hamiltonian

$$H=\frac{1}{2m}(\mathbf{p}-\mu\mathbf{V})^2+\frac{\alpha}{r}\,, \quad (6.1)$$

with

$$\mathbf{V}=\mathbf{n}\cdot\mathbf{x}\frac{\mathbf{n}\times\mathbf{x}}{|\mathbf{n}\times\mathbf{x}|^2} \quad (6.2)$$

and $\alpha=(e_1e_2+g_1g_2)/4\pi$, $\mu=(e_1g_2-g_1e_2)/4\pi c$. The curl of the vector potential $\mathbf{V}$ gives the radial Coulombic field plus a spurious field with support on the Dirac string $\mathbf{x}=\hat{\mathbf{n}}|\mathbf{x}|$,

$$\nabla\times\mathbf{V}=[\mathbf{x}/r^3-4\pi\mathbf{n}(\mathbf{n}\cdot\nabla)^{-1}(\mathbf{x})]\,, \quad (6.3)$$

where $(\mathbf{n}\cdot\nabla)^{-1}(x)$ is defined by Eq. (5.16). The operator

$$\mathbf{J}\equiv\mathbf{x}\times(\mathbf{p}-\mu\mathbf{V})-\mu\hat{x}\,, \quad (6.4)$$

with $\hat{x}=\mathbf{x}/r$, which corresponds to the classical expression for the angular momentum, satisfies the commutation relations

$$[\mathbf{J},H]=-i2\pi(\mu/\mathbf{n}^2)\{\mathbf{n}\times[\mathbf{n}\times(\mathbf{p}-\mu\mathbf{V})]\mathbf{n}\cdot\mathbf{x}(\mathbf{n}\cdot\nabla)^{-1}(x)$$
$$+\mathbf{n}\cdot\mathbf{x}(\mathbf{n}\cdot\nabla)^{-1}(x)\mathbf{n}\times[\mathbf{n}\times(\mathbf{p}-\mu\mathbf{V})]\}, \quad (6.5)$$

$$[J_i,J_j]=i\epsilon_{ijk}[J_k-(4\pi\mu/\mathbf{n}^2)n_k(\mathbf{n}\cdot\mathbf{x})^2(n\cdot\nabla)^{-1}(x)]. \quad (6.6)$$

Thus $\mathbf{J}$ would commute with H and would have the commutation relations of an angular momentum operator, except for terms corresponding to the spurious field, whose support is on the Dirac string, $\mathbf{x}=\pm\hat{n}|\mathbf{x}|$. However, we have seen in our classical discussion that the Lagrangian equations of motion are correct only when the Dirac string drawn from one particle does not go through the other particle. When translated into the quantum-mechanical language of Hilbert space, this condition becomes a restriction on the domain of definition of H and $\mathbf{J}$ which is thereby limited to functions that vanish sufficiently rapidly on the Dirac string. It is found that H and $\mathbf{J}$ may then be extended to self-adjoint operators satisfying $[H,\mathbf{J}]=0$, $[J_i,J_j]$

[16] C. A. Hurst, Ann. Phys. (N. Y.) 50, 51 (1968).

$=i\epsilon_{ijk}J_k$, namely, the commutation relations of the Lie algebra of the rotation group, with H an invariant. This is true for any value of μ. However, if one further requires that the J_i generate a representation of the finite rotation group and not just the infinitesimal Lie algebra, then μ becomes quantized and only the values $\mu=0, \pm1, +2\ldots$ are allowed.[16]

Let us now outline the same procedure in the field-theoretic case. The domain of definition of the operator is restricted to states $|\ \rangle$ satisfying

$$[(n\cdot\partial)^{-1}j_e\wedge(n\cdot\partial)^{-1}j_g]\,|\ \rangle=0. \tag{6.7}$$

On such states $n\wedge\partial_n\mathfrak{L}$ of Eq. (4.19) vanishes, and hence also the spurious last term of $\theta^{\mu\nu}$, Eq. (5.12), so $\theta^{\mu\nu}$ is symmetric and hence, by Eq. (4.26), $M^{\mu\nu}$ is conserved. Let us assume by analogy with the nonrelativistic case that the operators P^μ and $M^{\mu\nu}$ may be extended to self-adjoint operators satisfying the Lie algebra of the Poincaré group, and see what the consequences are if we also assume that they generate finite Poincaré transformations.

We begin by calculating the equal-time commutator of $M^{\mu\nu}$ with the dynamical variables ψ_n, A^μ, and B^μ, for this will give their infinitesimal transformation law. From Eqs. (4.25), (5.24), and (5.26), we find, after some algebra,

$$-i[\psi_n,M^{\mu\nu}]=[x^\mu\partial^\nu-x^\nu\partial^\mu+\tfrac{1}{4}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)+i\chi_n^{\mu\nu}]\psi_n\,, \tag{6.8a}$$

$$-i[A^\kappa,M^{\mu\nu}]=(x^\mu\partial^\nu-x^\nu\partial^\mu)A^\kappa+(g^{\kappa\mu}A^\nu-g^{\kappa\nu}A^\mu)-\partial^\kappa\chi_e^{\mu\nu}-(n\cdot\partial)^{-2}[n^\mu(n\wedge j_g)^{d\nu\kappa}-n^\nu(n\wedge j_g)^{d\mu\kappa}]\,, \tag{6.8b}$$

$$-i[B^\kappa,M^{\mu\nu}]=(x^\mu\partial^\nu-x^\nu\partial^\mu)B^\kappa+(g^{\kappa\mu}B^\nu-g^{\kappa\nu}B^\mu)-\partial^\kappa\chi_g^{\mu\nu}+(n\cdot\partial)^{-2}[n^\mu(n\wedge j_e)^{d\nu\kappa}-n^\nu(n\wedge j_e)^{d\mu\kappa}]\,, \tag{6.8c}$$

where

$$\chi_e^{\mu\nu}=(n\cdot\partial)^{-1}(n\wedge A)^{\mu\nu}\,,\quad \chi_g^{\mu\nu}=(n\cdot\partial)^{-1}(n\wedge B)^{\mu\nu}\,,$$
$$\chi_n^{\mu\nu}=e_n\chi_e^{\mu\nu}+g_n\chi_g^{\mu\nu}\,.$$

We see that apart from the spinor and vector transformations of ψ_n, A^μ, and B^μ and a gauge transformation χ which is not unexpected, there are extra terms involving the currents, which are added to the vector potentials.

Some quantities do transform simply, however. Thus $j_n{}^\kappa=\bar\psi_n\gamma^\kappa\psi_n$ transforms like a four-vector,

$$-i[j_n{}^\kappa,M^{\mu\nu}]=(x^\mu\partial^\nu-x^\nu\partial^\mu)j_n{}^\kappa+g^{\kappa\mu}j_n{}^\nu-g^{\kappa\nu}g_n{}^\mu\,. \tag{6.9}$$

Contracting Eqs. (6.8b) and (6.8c) with n_κ we see that the gauge transformation χ makes the free fields $n\cdot A$ and $n\cdot B$ transform like scalars:

$$-i[n\cdot A,\,M^{\mu\nu}]=(x^\mu\partial^\nu-x^\nu\partial^\mu)n\cdot A\,,\quad\text{and}\quad A\to B\,. \tag{6.10}$$

Finally, we note that the extra terms involving the currents ensure that the electromagnetic field $F^{\kappa\lambda}$ transforms like a tensor:

$$-i[F^{\kappa\lambda},M^{\mu\nu}]=(x^\mu\partial^\nu-x^\nu\partial^\mu)F^{\kappa\lambda}+(g^{\kappa\mu}F^{\nu\lambda}-g^{\kappa\nu}F^{\mu\lambda})+(g^{\lambda\mu}F^{\kappa\nu}-g^{\lambda\nu}F^{\kappa\mu})\,. \tag{6.11}$$

This is most easily verified from Eq. (2.3) by writing $\partial^\kappa A^\lambda=-i[A^\lambda,P^\kappa]$ and making use of Jacobi's identity and the commutation relations of the Lie algebra of the Poincaré group.

Let us see what happens when we integrate these infinitesimal transformations to finite Lorentz transformations. The simple scalar, vector, and tensor transformations of $n\cdot A$, $n\cdot B$, $j_n{}^\mu$, and $F^{\mu\nu}$ are easily integrated. With finite Lorentz transformation $\Lambda=\exp(\Omega)$ and representative $U(\Lambda)=\exp(iM\cdot\Omega)$, $M\cdot\Omega\equiv\frac{1}{2}M^{\mu\nu}\Omega_{\nu\mu}$, we have

$$n\cdot A^\Lambda(x)=U(\Lambda)n\cdot A(x)U^{-1}(\Lambda)=n\cdot A(\Lambda x)\quad\text{and}\quad A\to B\,, \tag{6.12a}$$

$$j_n{}^{\Lambda\mu}(x)=U(\Lambda)j_n{}^\mu(x)U^{-1}(\Lambda)=\Lambda^{-1\mu}{}_\nu j_n{}^\nu(\Lambda x)\,, \tag{6.12b}$$

$$F^{\Lambda\mu\nu}(x)=U(\Lambda)F^{\mu\nu}(x)U^{-1}(\Lambda)=\Lambda^{-1\mu}{}_\sigma\Lambda^{-1\nu}{}_\tau F^{\sigma\tau}(\Lambda x)\,. \tag{6.12c}$$

These transformations and Eqs. (2.3) and (2.4) which relate the field and potentials determine the finite transformation laws of A and B,

$$A^{\Lambda\mu}(x)=U(\Lambda)A^\mu(x)U^{-1}(\Lambda)=\Lambda^{-1\mu}{}_\nu A^\nu(\Lambda x)+\partial^\mu\chi_e{}^\Lambda(x)+[(n\cdot\partial)(\Lambda^{-1}n\cdot\partial)]^{-1}\times\epsilon^\mu{}_{\rho\sigma\tau}n^\rho(\Lambda^{-1}n)^\sigma j_g{}^{\Lambda\tau}\,, \tag{6.13a}$$

$$B^{\Lambda\mu}(x)=U(\Lambda)B^\mu(x)U^{-1}(\Lambda)=\Lambda^{-1\mu}{}_\nu B^\nu(\Lambda x)+\partial^\mu\chi_g{}^\Lambda(x)-[(n\cdot\partial)(\Lambda^{-1}n\cdot\partial)]^{-1}\times\epsilon^\mu{}_{\rho\sigma\tau}n^\rho(\Lambda^{-1}n)^\sigma j_e{}^{\Lambda\tau}\,, \tag{6.13b}$$

where

$$\chi_e{}^\Lambda(x)=\int(n\cdot\partial)^{-1}(x-y)(n-\Lambda n)\cdot A(\Lambda y)d^4y\,, \tag{6.14a}$$

$$\chi_g{}^\Lambda(x)=\int(n\cdot\partial)^{-1}(x-y)(n-\Lambda n)\cdot B(\Lambda y)d^4y\,. \tag{6.14b}$$

The kernel $(n\cdot\partial n'\cdot\partial)^{-1}$ appearing in Eqs. (6.13) is ambiguous. If we restrict its support to the $n-n'$ plane and require that $n^2(n\cdot\partial)^{-2}(x)=n\cdot x(n\cdot\partial)^{-1}(x)$, which was used in deriving Eqs. (6.8), then we have

$$(n\cdot\partial n'\cdot\partial)^{-1}=\frac{1}{4}\int\delta^4(x-ns-n't)[\epsilon(s)\epsilon(t)+1]dsdt\,. \tag{6.15}$$

When Λ is infinitesimal one easily verifies that Eqs. (6.13) reduces to Eqs. (6.8b) and (6.8c). To verify the group multiplication law, one makes use of the property

$$U(\Lambda_2)\chi^{\Lambda_1}(x)U^{-1}(\Lambda_2)=\chi^{\Lambda_{21}}(x)-\chi^{\Lambda_2}(\Lambda_1 x) -[(n\cdot\partial n_1\cdot\partial n_{21}\cdot\partial)^{-1}\epsilon_{\kappa\lambda\mu\nu}n^\kappa n_1{}^\lambda n_{21}{}^\mu j^{\Lambda_{21}}](x)\,, \quad (6.16)$$

where $\Lambda_{21}=\Lambda_2\Lambda_1$, $n_1=\Lambda_1{}^{-1}n$, and $n_{21}=\Lambda_{21}{}^{-1}n$.

The Lorentz transformation law for ψ is determined from Eqs. (6.8) and (6.13) by the general principle of gauge invariance,[17]

$$\psi^\Lambda(x)=U(\Lambda)\psi_n(x)U^{-1}(\Lambda)=S^{-1}(\Lambda)\psi_n(\Lambda x) \times\exp[-i\chi_n{}^\Lambda(x)]\,, \quad (6.17)$$

where $\chi_n{}^\Lambda=e_n\chi_e{}^\Lambda+g_n\chi_g{}^\Lambda$. The meaning of this expression must be made more precise because χ^Λ is an operator gauge function which does not commute with ψ in general. We recall that n^μ is assumed to have a vanishing time component $n^\mu=(0,\mathbf{n})$ and that we only know equal-time commutation relations. Referring back to the definition of χ^Λ, Eqs. (6.14), we observe that when Λ belongs to the little group of n, $(\Lambda n=n)$ then χ^Λ vanishes, so ψ_n transforms like a spinor. We also observe that if Λ is a pure rotation $(\Lambda=R)$, $\psi_n(Rx)$ commutes with $\chi^R(x)$. Hence Eq. (6.17) is unambiguous for rotations and Lorentz transformations which belong to the little group of n. However, any Lorentz transformation may be written as a product of these, and thus the transformation law of ψ_n becomes determined for any Λ. For definiteness let us suppose that n^μ lies along the three-axis, $n^\mu=(-n^2)^{1/2}(0,0,0,1)$. We decompose any Λ into the product

$$\Lambda=RV^{(1)}S\,, \quad (6.18)$$

where R and S are rotations and $V^{(1)}$ is a velocity transformation along the 1 axis and thus in the little group of n. Then for arbitrary Λ we have the transformation law

$$\psi_n{}^\Lambda(x)=S^{-1}(\Lambda)\psi_n(\Lambda x)\exp[-i\chi_n{}^R(V^{(1)}Sx)] \times\exp[-iU(RV^{(1)})\chi_n{}^S(x)U^{-1}(RV^{(1)})]\,. \quad (6.19)$$

We must now verify that this law has the group multiplication property. This is done in two steps. One first verifies that it holds for rotations, as will be described in detail below because it yields the charge-quantization condition. Once it is known that the group property holds for rotations, one verifies it for an arbitrary Lorentz transformation by performing on ψ the succession of operations which bring a product of Λ's of the form (6.18) back in this form. We indicate the order of these operations, without writing them explicitly, beginning with

$$\Lambda_2\Lambda_1=R_2V_2{}^{(1)}S_2R_1V_1{}^{(1)}S_1\,.$$

[17] One could presumably introduce an additional gauge transformation on ψ with additional gauge function χ given by the line integral of the last term of Eq. (6.13) (problems of operator ordering arise), even though it appears to have the wrong infinitesimal limit. For if the charge quantization condition (6.25) holds, then for a finite Lorentz transformation, the presumed additional phase factor $e^{i\chi}$ not only becomes single-valued, but in fact reduces to unity.

The product of rotations S_2R_1 may be decomposed as

$$S_2R_1=R_a{}^{(1)}R_b{}^{(3)}R_c{}^{(1)}\,,$$

where $R^{(i)}$ are rotations about the i axis; so, commuting $R^{(1)}$ with $V^{(1)}$, we have

$$\Lambda_2\Lambda_1=R_2R_a{}^{(1)}V_2{}^{(1)}R_b{}^{(3)}V_1{}^{(1)}R_c{}^{(1)}S_1\,.$$

We observe that $V^{(1)}$ and $R^{(3)}$ are in the little group of n^μ, so the rearrangement

$$V_2{}^{(1)}R_b{}^{(3)}V_1{}^{(1)}=R_d{}^{(3)}V_{21}{}^{(1)}R_e{}^{(3)}$$

is possible and yields

$$\Lambda_2\Lambda_1=[R_2R_a{}^{(1)}R_d{}^{(3)}]V_{21}{}^{(1)}[R_e{}^{(3)}R_c{}^{(1)}S_1]\,,$$

which is of the required form. The main work involved is to verify, by repeated use of (6.14) and (6.16), that the gauge factors χ^Λ combine properly.

We now verify explicitly that the group property holds for rotations. From Eqs. (6.16) and (6.17), we obtain

$$U(R_2)U(R_1)\psi_n(\mathbf{x})U^{-1}(R_1)U^{-1}(R_2) =S^{-1}(R_{21})\psi(R_{21}\mathbf{x})\exp[-i\chi_n{}^{R_2}(R_1\mathbf{x})] \times\exp\{-i[\chi_n{}^{R_{21}}(\mathbf{x})-\chi_n{}^{R_2}(R_1\mathbf{x})-\phi(\mathbf{x})]\}\,, \quad (6.20)$$

where

$$\phi=\mathbf{n}\cdot\mathbf{n}_1\times\mathbf{n}_{21}(\mathbf{n}\cdot\nabla\mathbf{n}_1\cdot\nabla\mathbf{n}_{21}\cdot\nabla)^{-1} \times(e_n\rho_g{}^{R_{21}}-g_n\rho_e{}^{R_{21}}) \quad (6.21)$$

and $R_{21}=R_2R_1$, $\mathbf{n}_1=R_1{}^{-1}\mathbf{n}$, $\mathbf{n}_{21}=R_{21}{}^{-1}\mathbf{n}$. $\phi(\mathbf{x})$ commutes with the other factors in the exponent and one may check that $\chi_n{}^{R_2}(R_1\mathbf{x})$ commutes with $\chi_n{}^{R_{21}}(\mathbf{x})$. Hence we have

$$U(R_2)U(R_1)\psi_n(\mathbf{x})U^{-1}(R_1)U^{-1}(R_2) =U(R_{21})\psi_n(\mathbf{x})U^{-1}(R_{21})e^{i\phi(\mathbf{x})} \quad (6.22)$$

and the group property is satisfied only if $e^{i\phi(\mathbf{x})}$ is unity, which means that the eigenvalues of the operator $\phi(\mathbf{x})$ must be integral multiples of 2π. To study the eigenvalues of $\phi(\mathbf{x})$ we commute it with $\psi_m{}^{R_{21}}(\mathbf{y})$ at equal times and find

$$[\phi(\mathbf{x}),\psi_m{}^{R_{21}}(\mathbf{y})]=-(e_ng_m-g_ne_m)\mathbf{n}\cdot\mathbf{n}_1 \times\mathbf{n}_{21}(\mathbf{n}\cdot\nabla\mathbf{n}_1\cdot\nabla\mathbf{n}_{21}\cdot\nabla)^{-1}(\mathbf{x}-\mathbf{y})\psi_m{}^{R_{21}}(\mathbf{y})\,. \quad (6.23)$$

The operator $(n\cdot\nabla\mathbf{n}_1\cdot\nabla\mathbf{n}_{21}\cdot\nabla)^{-1}$ is ambiguous but becomes determinate by requiring consistency with Eq. (6.15), and we find

$$\mathbf{n}\cdot\mathbf{n}_1\times\mathbf{n}_{21}(\mathbf{n}\cdot\nabla\mathbf{n}_1\cdot\nabla\mathbf{n}_{21}\cdot\nabla)^{-1}(\mathbf{x}) =\mathbf{n}\cdot\mathbf{n}_1\times\mathbf{n}_{21}\frac{1}{8}\int\delta^3(\mathbf{x}-\mathbf{n}s-\mathbf{n}_1t-\mathbf{n}_{21}u) \times[\epsilon(s)\epsilon(t)\epsilon(u)+\epsilon(s)+\epsilon(t)+\epsilon(u)]dsdtdu =\tfrac{1}{8}[\epsilon(s_0)\epsilon(t_0)\epsilon(u_0)+\epsilon(s_0)+\epsilon(t_0)+\epsilon(u_0)]\,, \quad (6.24)$$

where $s_0=\mathbf{x}\cdot\mathbf{n}\times\mathbf{n}_1$, $t_0=\mathbf{x}\cdot\mathbf{n}_1\times\mathbf{n}_{21}$, and $u_0=\mathbf{x}\cdot\mathbf{n}_{21}\times\mathbf{n}$. We see that $\psi_m^{R_{21}}(y)$ connects eigenstates of $\phi(x)$, whose eigenvalues differ by 0 or $\pm\frac{1}{2}(e_ng_m-g_ne_m)$ which must be an integral multiple of 2π or

$$(e_ng_m-g_ne_m)/4\pi=Z_{nm}\,, \tag{6.25}$$

where Z_{nm} is an integer. Since ψ_m, $\psi_m^\dagger$, A, and B form a complete set of operators, this is also a sufficient condition for $e^{i\phi(\mathbf{x})}=1$. Thus the requirement that the representation of the Lie algebra of the Poincaré group integrate to a representation of the finite group yields the quantization condition obtained earlier.[2,4,18]

If one wishes to verify that the equations of motion are invariant under the transformations given here, one encounters a paradox. For the invariance depends on use of the apparently contradictory relations[17]

$$e^{2\pi i\epsilon(\phi)}=1\,,\quad \frac{\partial}{\partial\phi}e^{2\pi i\epsilon(\phi)}=4\pi i\delta(\phi)\,.$$

A more explicit construction of the operator extension to replace the formal discussion presented here is required to resolve this paradox adequately.

VII. FEYNMAN RULES

An important question which remains to be answered is how Green's functions in general and S-matrix elements in particular transform. In an explicit nonrelativistic calculation[6] it was found that the amplitude for scattering of spinless particles does not transform like a scalar, but picks up an extra phase characteristic of helicity-flip amplitudes. It is important to find out the transformation law in the relativistic field-theoretic case to replace the conjecture presented earlier.[6] A likely candidate for study is the infinite series of Feynman diagrams represented by the T product

$$S=T\exp\left[i\int\mathcal{L}_I(x)d^4x\right], \tag{7.1}$$

where, from Eq. (4.4),

$$\mathcal{L}_I(x)=-(j_e\cdot A+j_g\cdot B)$$

and

$$j_e^\mu=\sum_n e_n\bar\psi_n\gamma^\mu\psi_n\,,\quad j_g^\mu=\sum_n g_n\bar\psi_n\gamma^\mu\psi_n\,.$$

Here ψ_n, A, and B satisfy the free-field equations. The justification for this expression is that

$$S=T\exp\left(-i\int H_Idt\right)$$

and that, by Eq. (5.24),

$$H_I=-L_I=-\int\mathcal{L}_Id^3x\,.$$

The charged particle propagators are as usual

$$-i\langle T\psi_n(x)\bar\psi_n(y)\rangle=(i\gamma\cdot\partial+m_n)\Delta_F(x-y,m_n)\,, \tag{7.2}$$

where $\Delta_F(x,m)$ is the Feynman propagator for a scalar particle of mass m, $(-\partial^2-m^2)\Delta_F(x,m)=\delta^4(x)$. The photon propagator presents new features. It is most easily expressed as a 2×2 matrix in charge space. Defining

$$V_\mu{}^a\equiv(A_\mu,B_\mu)\,,\quad (a=1,2) \tag{7.3}$$

we write the free Lagrangian equations of motion for A and B as matrix equations. From Eqs. (5.7), with vanishing sources, we have

$$(1/n^2)(n\cdot\partial)(n\cdot\partial V_\mu{}^a-\partial_\mu n\cdot V^a-n_\mu\partial\cdot V^a-\epsilon^{ab}\epsilon_\mu{}^{\nu\kappa\lambda}n_\nu\partial_\kappa V_\lambda{}^b)=0\,. \tag{7.4}$$

Here ϵ^{ab} is the antisymmetric two-dimensional tensor with $\epsilon^{12}=1$. The equal-time commutation relations (5.15) take the form

$$[V_\mu{}^a(t,\mathbf{x}),V_\nu{}^b(t,\mathbf{y})]=[i\epsilon^{ab}\epsilon_{\mu\nu\kappa0}n^\kappa-i\delta^{ab}(g_{\mu0}n_\nu+g_{\nu0}n_\mu)](\mathbf{n}\cdot\boldsymbol{\nabla})^{-1}(\mathbf{x}-\mathbf{y})\,. \tag{7.5}$$

Applying the Lagrangian differential operator (7.4) to the photon propagator $(-i)\langle TV_\mu{}^a(x)V_\nu{}^b(y)\rangle$, we obtain

$$(1/n^2)(n\cdot\partial)[(n\cdot\partial g_\lambda{}^\mu-\partial_\lambda n^\mu-n_\lambda\partial^\mu)\delta^{ab}-(\epsilon_{\lambda\sigma\tau}{}^\mu n^\sigma\partial^\tau)\epsilon^{ab}](-i)\langle TV_\mu{}^b(x)V_\nu{}^c(y)\rangle=\delta^{ac}g_{\lambda\nu}\delta^4(x-y)\,. \tag{7.6}$$

Thus, we see that as in other Lagrangian theories, the propagator is the Green's function for the Lagrangian differential operator. The solution to this equation with Feynman boundary conditions is easily verified to be[19]

$$-i\langle TV_\mu{}^a(x)V_\nu{}^b(y)\rangle=\{[-g_{\mu\nu}+(\partial_\mu n_\nu+n_\mu\partial_\nu)(n\cdot\partial)^{-1}]\delta^{ab}-[\epsilon_{\mu\nu\sigma\tau}n^\sigma\partial^\tau(n\cdot\partial)^{-1}]\epsilon^{ab}\}\Delta_F(x-y,0)\,. \tag{7.7}$$

The Feynman rules are completed by specifying that the vertex $e\gamma^\mu$ of ordinary electrodynamics is replaced for the nth charged particle by $q_n{}^a\gamma^\mu$, where the charge vector $q_n{}^a$ is given by

$$q_n{}^a=(e_n,g_n)\,. \tag{7.8}$$

It is contracted with the photon line entering or leaving the vertex. We observe that the charges only appear

[18] Schwinger (Ref. 2) asserts that one should assign the median value to a step function at its point of discontinuity and that consideration of this point requires an extra factor of 2 in the charge-quantization condition.

[19] The off-diagonal part of the propagator already appears in the literature: B. Zumino, Ref. 13; S. Weinberg, Phys. Rev. **138**, B988 (1965); J. G. Taylor, in *Lectures in High Energy Physics*, edited by H. H. Aly (Wiley, New York, 1968); A. Rabl, Phys. Rev. **179**, 1363 (1969).

in the combinations $q_n{}^a\delta^{ab}q_m{}^b$ and $q_n{}^a\epsilon^{ab}q_m{}^b$ which are invariant under rotations in the two-dimensional charge space, so the Feynman diagrams individually manifest chiral invariance discussed previously.[4] If all charge vectors are parallel, then only the δ^{ab} term contributes to the photon propagator and ordinary electrodynamics in a particular gauge is recovered.

The infinite series of Feynman diagrams representing the S operator (7.1) is not useful for practical calculations because of the large magnetic coupling constant $g^2/4\pi \approx (137)n$ implied by the charge-quantization condition, but its formal properties may be of interest. For example, one might hope to deduce from it the behavior of the scattering amplitude under Lorentz transformation. However, even this would not be simple, for the group property is not satisfied order by order [as is obvious from the transformation law (6.17)], and when the charge quantization condition holds, the same power of the charge appears in an infinite number of diagrams of different order. (By "order" one means here the number of vertices in a Feynman diagram.) We hope to return to the transformation law of Green's functions on another occasion.

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Kinematic Superstructures for Abnormal Couplings in the 8-π Dual Amplitude*

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We present a general method for including abnormal couplings in multiparticle dual amplitudes. We construct amplitudes from a sum of terms having kinematic superstructure and dual substructure; and we show how a tensorial analysis of the kinematic superstructure in various channels determines the normality of the trajectories involved. As an illustration, we give a specific solution for abnormal couplings in the 8-π amplitude in which the above analysis is carried out in detail, and is compared with previous 6-π amplitudes and four-point amplitudes for spin-1–spin-0 scattering.

I. INTRODUCTION

ANALYSES of the structure of the N-scalar-particle dual amplitude by Chan *et al.*[1] and by Koba and Nielsen[2] have shown that it predicts normal coupling at all three-point vertices. [A coupling is normal or abnormal if the product of the normalities of the three particles is $+1$ or -1. A particle is normal, $n=+1$, if it has parity $(-1)^J$ or abnormal, $n=-1$, if it has parity $-(-1)^J$.] The existence of abnormal vertices is essentially a complication due to spin; any vertex with two spinless particles conserves normality. The problem of choosing the normality of an internal trajectory thus first arises in the four-point functions in reactions like $\rho\pi \to \rho\pi$; the problem already occurs in the 3-π trajectories of the 6-π amplitude. We reexamine the previous analyses of four-point[3–7] and six-point[8,9] amplitudes which are concerned with prescribing the normalities of internal trajectories. From this analysis we are able to propose a procedure for writing amplitudes for N-π's with defined leading normality on internal trajectories. In particular we concern ourselves with having the ω-A_2 trajectory in certain 3-π channels of the 8-π amplitude.

For four-point amplitudes with external spinning particles, one usually writes the invariant Lorentz tensors contracted against helicities and makes use of the analysis of Gell-Mann *et al.*[10] to determine the normalities of the invariant amplitudes associated with them. This method is clearly impractical when we come to analyze processes involving high spins. This is especially true since invariant amplitudes tend to give normal couplings. The method we adopt involves the use of "noninvariant" amplitudes, whose use has been de-

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